PMM U.S.S.R.,Vol.54,No.6,pp.825-832,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

NON-LINEAR FREE SPATIAL VIBRATIONS OF COMBINED SUSPENSION SYSTEMS*

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Free vibrations of a combined suspension system with a beam with stiffness which has one axis of symmetry are studied. Unlike /1/ the case of internal resonance in the system is analysed in detail. It is shown that both in this case and in the case where the frequencies of the vertical modes and the bending and twisting modes are close to each other /1/ there are two possible forms of vibrations, namely, those which are subject to phase and amplitude modulation and are accompanied by energy transmission, and those subject to phase modulation only, which allow no energy transmission.

1. In the general case of a deformed state of the suspension system (Fig.1) with shear deformations, the inertia of rotation of the cross-section of the stiff beam, and the inertia of the carrying cables neglected, the non-linear equations of free bending and twisting vibrations in dimensionless form can be written as follows /2/:

$$\begin{gathered}\eta^{\cdot\cdot} + I_x\eta'''' - 2H_q\eta'' + {}^1/_2 S_k q^2 \int \eta dz - S_k q\, [\eta'' \int \eta dz + \\ \varphi'' \int \varphi dz + {}^1/_2 \int (\eta'^2 + \varphi'^2)\, dz] - S_k\, [\eta'' \int (\eta'^2 + \\ \varphi'^2)\, dz - 2\varphi'' \int \eta'\varphi' dz] = 0 \\ \varphi^{\cdot\cdot} + {}^1/_4 a_y b^2 s^{\cdot\cdot} + I_\omega \varphi'''' - [GI_k + 2\,(c_y^2 + {}^1/_4 b^2)\, H_q]\, \varphi'' + \\ {}^1/_8 b^2 q^2 S_k \int \varphi dz - {}^1/_4 b^2 q S_k\, [\varphi'' \int \eta dz + \eta'' \int \varphi dz - \\ \int \eta'\varphi' dz] - {}^1/_4 b^2 S_k\, [\varphi'' \int (\eta'^2 + \varphi'^2)\, dz + 2\eta'' \int \eta'\varphi' dz] = 0 \\ s^{\cdot\cdot} + a_y \varphi^{\cdot\cdot} + I_y s'''' + qh^{-1}s = 0\end{gathered} \tag{1.1}$$

Here q is the weight of the bridge per unit length, E, G, I_x, I_y, I_k, and I_ω are the moduli of elasticity of the first and second kind, the moments of inertia with respect to the central axes x and y, the twisting moment of inertia, and the sectorial moment of intertia, respectively, H_q is the thrust due to a constant load in the parabolically shaped cable, b is the width of the stiff beam, c_y is the distance from the centre of mass of the cross-section to the horizontal median line of the cross-section, a_y is the coordinate of the centre of bending A, S_k is the stiffness of the cable, s and η are the displacements of the bending centre $A\,(0, a_y)$ along the x and y directions, φ is the angle of rotation of the cross-section around the z axis, differentiation with respect to the spatial coordinate z is denoted by an apostrophe, differentiation with respect to the time variable is denoted by a dot, and integration is always carried out over the length L of the stiff beam.

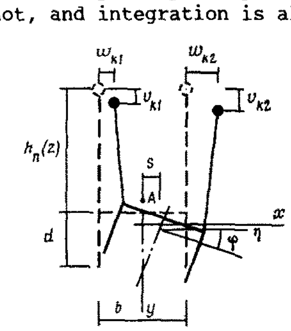

Fig.1

To derive (1.1), the dimensional variables z, s and η were divided by the wavelength λ of the highest mode of vibrations taken into account, the time t was divided by $\sqrt{q\,(Eg)^{-1}}$ (g is the free fall acceleration), and the dynamical thrust was defined as in /1/ taking the non-linear terms into account.

We shall seek a solution of (1.1) of the form (the sums are from $m = 1$ to $m = \infty$)

$$\begin{gathered}\eta\,(z, t) = \Sigma v_m\,(z)\, x_{1m}\,(t), \quad \varphi\,(z, t) = \Sigma \theta_m\,(z)\, x_{2m}\,(t) \\ s\,(z, t) = \Sigma u_m\,(z)\, x_{2m}\,(t)\end{gathered} \tag{1.2}$$

Here x_{1m} and x_{2m} are the generalized displacements, v_m, θ_m and u_m are the vertical, twisting, and horizontal linear characteristic modes of vibrations of the suspension bridge, respectively, which satisfy the orthogonality condition

$$(p_i^2 - p_j^2) \int [\theta_i\theta_j + {}^1/_4 b^2 u_i u_j + {}^1/_4 a_y b^2\,(u_i\theta_j + u_j\theta_i) + v_i v_j]\, dz = 0 \tag{1.3}$$

*Prikl.Matem.Mekhan.,54,6,1003-1011,1990 where p_i can be equal to the characteristic frequencies of bending and twisting vibrations ω_{0i} and Ω_{0i}.

Substituting (1.2) into the resolving Eq.(1.1) and taking (1.3) into account, we obtain a system of non-linear equations in $x_{1i}(t)$ and $x_{2i}(t)$, whose form is identical with the analogous system given in /1/.

If there are only two prevailing modes in the process of vibrations, namely, the n-th mode of vertical vibrations and the m-th mode of bending and twisting vibrations (the interaction between the two modes can be observed either in the case where the frequencies of the modes are close to each other or in the case of internal resonance, i.e., for $\omega_{0n} \approx 2\Omega_{0m}$), then the resolving system of equations takes the form

$$\begin{gathered}
\ddot{x}_{1n} + \omega_{0n}^2 x_{1n} + a_{11}{}^n x_{1n}^2 + a_{22}^{nm} x_{2m}^2 + (b_{11}{}^n x_{1n}^2 + b_{22}^{nm} x_{2m}^2) x_{1n} = 0 \\
\ddot{x}_{2m} + \Omega_{0m}^2 x_{2m} + a_{12}^{nm} x_{1n} x_{2m} + (c_{11}^{nm} x_{1n}^2 + c_{22}{}^m x_{2m}^2) x_{2m} = 0 \\
\mu_n a_{11n} = {}^3/{}_2 S_k q \int v_n\, dz \int v_n'^2\, dz, \quad \mu_n a_{22}^{nm} = 2\lambda_m b^{-2} a_{12}^{nm} = \\
S_k q \left[\int \theta_m\, dz \int \theta_m' v_n'\, dz + {}^1/{}_2 \int \theta_m'^2\, dz \int v_n\, dz\right] \\
\mu_n b_{11}{}^n = S_k [\int v_n'^2 dz]^2, \quad \mu_n b_{22}{}^{nm} = 4\lambda_m b^{-2} c_{11}{}^{nm} = S_k [\int \theta_m'^2 dz \times \\
\int v_n'^2 dz + 2 (\int \theta_m' v_n' dz)^2], \quad 4\lambda_m b^{-2} c_{22}{}^m = S_k [\int \theta_m'^2 dz]^2 \\
\mu_n = \int v_n^2 dz, \; \lambda_m = \int (\theta_m^2 + {}^1/{}_4 b^2 u_m^2 + {}^1/{}_2 a_y b^2 u_m \theta_m)\, dz
\end{gathered} \tag{1.4}$$

In what follows the indices n and m will be omitted for simplicity.

For small but finite amplitudes, the solution of the system of Eqs.(1.4) can be represented by means of an expansion in various times scales in the form /3/

$$\begin{gathered}
x_\alpha(t) = \varepsilon x_{\alpha 1} + \varepsilon^2 x_{\alpha 2} + \varepsilon^3 x_{\alpha 3} + \ldots; \; x_{\alpha\beta} = x_{\alpha\beta}(T_0, T_1, T_2), \\
\alpha = 1, 2; \quad \beta = 1, 2, \ldots
\end{gathered} \tag{1.5}$$

where ε is a small parameter of the order of the amplitudes and $T_n = \varepsilon^n t \; (n = 0, 1, 2, \ldots)$ serve as new independent variables.

Taking into account that

$$\begin{gathered}
d/dt = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots, \; d^2/dt^2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + \\
2 D_0 D_2) + \ldots
\end{gathered}$$

substituting (1.5) into (1.4), and equating to zero the coefficients of ε, ε^2, and ε^3, we find that

$$D_0^2 x_{\alpha 1} + \zeta_\alpha^2 x_{\alpha 1} = 0, \quad \zeta_1 = \omega_0, \quad \zeta_2 = \Omega_0 \tag{1.6}$$

$$\begin{gathered}
D_0^2 x_{12} + \zeta_1^2 x_{12} = -2 D_0 D_1 x_{11} - a_{11} x_{11}^2 - a_{22} x_{21}^2 \\
D_0^2 x_{22} + \zeta_2^2 x_{22} = -2 D_0 D_1 x_{21} - a_{12} x_{11} x_{21}
\end{gathered} \tag{1.7}$$

$$\begin{gathered}
D_0^2 x_{13} + \zeta_1^2 x_{13} = -2 D_0 D_1 x_{12} - (D_1^2 + 2 D_0 D_2)\, x_{11} - \\
2 a_{11} x_{11} x_{12} - 2 a_{22} x_{21} x_{22} - b_{11} x_{11}^3 - b_{22} x_{21}^2 x_{11} \\
D_0^2 x_{23} + \zeta_2^2 x_{23} = -2 D_0 D_1 x_{22} - (D_1^2 + 2 D_0 D_2)\, x_{21} - \\
a_{12} (x_{11} x_{22} + x_{12} x_{21}) - c_{11} x_{11}^2 x_{21} - c_{22} x_{21}^3
\end{gathered} \tag{1.8}$$

We will seek a solution of (1.6) in the form

$$x_{\alpha 1} = A_\alpha (T_1, T_2) \exp(i \zeta_\alpha T_0) + \bar{A}_\alpha (T_1, T_2) \exp(-i \zeta_\alpha T_0) \tag{1.9}$$

where A_α are unknown complex functions and $\bar{A}_\alpha$ are the corresponding complex conjugate functions.

Substituting (1.9) into (1.7) we find that

$$\begin{gathered}
D_0^2 x_{12} + \zeta_1^2 x_{12} = -2 i \zeta_1 D_1 A_1 \exp(i \zeta_1 T_0) - a_{11} [A_1^2 \exp(2 i \zeta_1 T_0) + \\
A_1 \bar{A}_1] - a_{22} [A_2^2 \exp(2 i \zeta_2 T_0) + A_2 \bar{A}_2] + \mathrm{cc} \\
D_0^2 x_{22} + \zeta_2^2 x_{22} = -2 i \zeta_2 D_1 A_2 \exp(i \zeta_2 T_0) - \\
a_{12} \{A_1 A_2 \exp[i (\zeta_1 + \zeta_2) T_0] + A_1 \bar{A}_2 \exp[i (\zeta_1 - \zeta_2) T_0]\} + \mathrm{cc}
\end{gathered} \tag{1.10}$$

where cc denote the complex conjugate parts of the preceding terms.

2. We will consider the case of internal resonance, which occurs for $\zeta_1 \approx 2\zeta_2$. We will first assume that there are either two symmetric forms of vibrations or a symmetric form of vertical vibrations and a skew-symmetric form of bending and twisting vibrations taking part in the resonance. This implies that the coefficients a_{12} and a_{22} in (1.4) are non-zero. Since in this case the non-linearity of the equations determining the amplitudes manifests itself at the first step, it follows that to construct the solution, it suffices to restrict ourselves to terms of order ε^2. As a result, we arrive at relations (1.9) and Eqs.(1.10) (Eqs.(1.8) are not used), in which A_α are functions of T_1 only.

Taking into account that $\zeta_1 = 2\zeta_2 + \varepsilon\sigma$, where σ is the frequency detuning, we consider the expressions on the right-hand sides of Eqs.(1.10).

The functions $\exp(i\zeta_1 T_0)$ and $\exp(i\zeta_2 T_0)$ appearing in these expressions generate the secular terms, and so the coefficients of these functions must be equal to zero. As a result, we get

$$2i\zeta_1 D_1 A_1 + a_{22} A_2^2 \exp(-i\sigma T_1) = 0,$$
$$2i\zeta_2 D_1 A_2 + a_{12} A_1 \bar{A}_2 \exp(i\sigma T_1) = 0 \tag{2.1}$$

In (2.1) we replace the functions A_α by $A_\alpha \exp(i\sigma T_1)$, respectively. Then the expressions $\exp(\pm i\sigma T_1)$ disappear from the equations. We multiply the first equation by $\bar{A}_1$ and the second by $\bar{A}_2$, and we find the conjugate equations. First, we add the two pairs of mutually conjugate equations to each other and we then subtract one from another. As a result of the procedure described above, expressing A_α in the polar form $A_\alpha = a_\alpha \exp(i\varphi_\alpha)$, we arrive at the following system of four differential equations in a_α and φ_α:

$$\zeta_1 a_1^{2\cdot} + a_{22} a_2^2 a_1 \sin\gamma = 0, \quad \zeta_2 a_2^{2\cdot} - a_{12} a_2^2 a_1 \sin\gamma = 0 \tag{2.2}$$
$$\varphi_1^{\cdot} + \sigma - {}^1/_2 a_{22} \zeta_1^{-1} a_2^2 a_1^{-1} \cos\gamma = 0$$
$$\varphi_2^{\cdot} + \sigma - {}^1/_2 a_{12} \zeta_2^{-1} a_1 \cos\gamma = 0 \quad (\gamma = 2\varphi_2 - \varphi_1)$$

where the dots denote differentiation with respect to T_1.

From the first two equations of system (2.2) we find that

$$a_1 = \sqrt{W\xi}, \quad a_2 = \sqrt{a_{12}\zeta_1 (a_{22}\zeta_2)^{-1} W (1-\xi)} \tag{2.3}$$
$$\xi^{\cdot} = -a_{12}\zeta_2^{-1}(1-\xi)\sqrt{W\xi}\sin\gamma \quad (W = a_1^2 + a_{22}\zeta_2 (a_{12}\zeta_1)^{-1} a_2^2)$$

where W is the energy of the system.

From the third and fourth equations of (2.2) we get

$$\gamma^{\cdot} = -{}^1/_2 a_{12}\zeta_2^{-1} W (1 - 3\xi)(W\xi)^{-1/2}\cos\gamma - \sigma$$

Using the expression

$$\gamma^{\cdot} = \frac{d\gamma}{d\xi}\xi^{\cdot} = a_{12}\zeta_2^{-1}(1-\xi)\sqrt{W\xi}\,\frac{d\cos\gamma}{d\xi}$$

for the derivative and integrating the equation, we find that

$$\cos\gamma = -\frac{\sigma\zeta_2}{a_{12}\sqrt{W}}\frac{\sqrt{\xi}}{(1-\xi)} + \frac{G}{\sqrt{\xi}(1-\xi)} \tag{2.4}$$

where G is an arbitrary constant, which can be determined from the initial conditions.

Expression (2.4) determines the amplitude-versus-phase dependence for the vibrations subjected both to phase and amplitude modulation and described by Eqs.(2.2).

To determine time dependence of the amplitudes and phases, it is necessary to substitute (2.4) into the third relation in (2.3), from which one can find ξ as a function of T_1 by integrating the equation. Using the function $\xi(T_1)$ in the first two formulae in (2.3) and in Eqs.(2.2), we find $a_\alpha(T_1)$ and $\varphi_\alpha(T_1)$.

We will now assume that the resonance involves either two skew-symmetric forms of vibrations or an asymmetric form of vertical vibrations along with a symmetric form of bending and twisting vibrations. The coefficients a_{11}, a_{12}, and a_{22} in (1.4) are equal to zero. In this case the system of Eqs.(2.1) yields $D_1 A_\alpha = 0$, i.e., A_α are independent of T_1. To determine the dependence of A_α on T_2 it is necessary to invoke the terms of order ε^3 and use the system of Eqs.(1.8). To avoid any secular terms which may appear in the solution of this system, one should equate the coefficients of $\exp(\pm i\zeta_\alpha T_2)$ to zero. Hence, we obtain the system of equations

$$2i\zeta_1 A_1^{\cdot} + 3b_{11}A_1^2\bar{A}_1 + 2b_{22}A_2\bar{A}_2A_1 = 0$$
$$2i\zeta_2 A_2^{\cdot} + 2c_{11}A_1\bar{A}_1A_2 + 3c_{22}A_2^2\bar{A}_2 = 0$$

Repeating for this system the operations which were applied to the system of Eqs.(2.1), we get

$$a_\alpha = \text{const}, \quad \varphi_1^{\cdot} = \omega_1 = {}^3/_2 b_{11}\zeta_1^{-1}a_1^2 + b_{22}\zeta_1^{-1}a_2^2 \qquad (2.5)$$
$$\varphi_2^{\cdot} = \omega_2 = c_{11}\zeta_2^{-1}a_1^2 + {}^3/_2 c_{22}\zeta_2^{-1}a_2^2$$

where the dots denote differentiation with respect to T_2.

On the basis of (2.5) the expressions for the generalized displacements can be written in the form

$$x_\alpha(t) = 2a_\alpha\varepsilon\cos[(\zeta_\alpha + \varepsilon^2\omega_\alpha)t + \varphi_{\alpha 0}] + O(\varepsilon^3)$$

where $\varphi_{\alpha 0}$ are the initial phases.

3. For comparison we will consider the vibrations in the case of two modes of vertical vibrations and bending and twisting vibrations with similar frequencies interacting with each other, i.e., in the case where

$$\omega_0 = \Omega_0 + \sigma\varepsilon^2 \qquad (3.1)$$

To analyse this case, we shall use the systems of Eqs.(1.6)-(1.8). In order that the solutions satisfying these systems do not contain any secular terms, it is necessary that the relations

$$-iA_1^{\cdot} - \lambda_1A_1^2\bar{A}_1 - \lambda_2A_1A_2\bar{A}_2 + {}^1/_4\Gamma_1\bar{A}_1A_2^2\exp(-2i\sigma T_2) = 0 \qquad (3.2)$$
$$-iA_2^{\cdot} - \lambda_3A_1\bar{A}_1A_2 - \lambda_4A_2^2\bar{A}_2 + {}^1/_4\Gamma_2A_1^2\bar{A}_2\exp(2i\sigma T_2) = 0$$

$$\lambda_1 = \frac{3b_{11}}{2\omega_0} - \frac{5a_{11}^2}{3\omega_0^3}, \quad \lambda_2 = \frac{b_{22}}{\omega_0} - \frac{2a_{11}a_{22}}{\omega_0^3} - \frac{2a_{12}a_{22}}{3\omega_0\Omega_0^2}$$
$$\lambda_3 = \frac{c_{11}}{\Omega_0} - \frac{a_{12}^2}{3\Omega_0^3} - \frac{a_{11}a_{12}}{\Omega_0\omega_0^2}, \quad \lambda_4 = \frac{3c_{22}}{2\Omega_0} - \frac{5a_{12}a_{22}}{6\Omega_0\omega_0^2}$$
$$\Gamma_1 = \frac{4a_{12}a_{22}}{\omega_0\Omega_0^2} - \frac{4a_{11}a_{22}}{3\omega_0^3} - \frac{2b_{22}}{\omega_0}, \quad \Gamma_2 = \frac{2a_{12}^2}{\Omega_0^3} - \frac{2a_{11}a_{12}}{3\omega_0^2\Omega_0} - \frac{2c_{11}}{\Omega_0}$$

be satisfied. Here A_α depend on T_2 and the dots denote differentiation with respect to T_2.

If A_2 is replaced by $A_2\exp(i\sigma T_2)$ in Eqs.(3.2), then the expressions $\exp(\pm 2i\sigma T_2)$ disappear. For the resulting system we repeat the operations which were applied in the case of the system of Eqs.(2.1) to find that

$$(a_1^2)^{\cdot} - {}^1/_2\Gamma_1a_1^2a_2^2\sin 2\gamma = 0, \quad (a_2^2)^{\cdot} + {}^1/_2\Gamma_2a_1^2a_2^2\sin 2\gamma = 0 \qquad (3.3)$$
$$\varphi_1^{\cdot} - \lambda_1a_1^2 - \lambda_2a_2^2 + {}^1/_4\Gamma_1a_2^2\cos 2\gamma = 0 \quad (\gamma = \varphi_2 - \varphi_1)$$
$$\varphi_2^{\cdot} - \lambda_3a_1^2 - \lambda_4a_2^2 + {}^1/_4\Gamma_2a_1^2\cos 2\gamma + \sigma = 0$$

From (3.3) we have

$$a_1 = \sqrt{W\xi}, \quad a_2 = \sqrt{\Gamma_2\Gamma_1^{-1}W(1-\xi)} \qquad (3.4)$$
$$\xi^{\cdot} = {}^1/_2\Gamma_2W\xi(1-\xi)\sin 2\gamma \quad (W = a_1^2 + \Gamma_1\Gamma_2^{-1}a_2^2) \qquad (3.5)$$
$$\gamma^{\cdot} = {}^1/_4\Gamma_2W(1-2\xi)\cos 2\gamma - (\lambda_1 - \lambda_3)W\xi -$$
$$\Gamma_2\Gamma_1^{-1}(\lambda_2 - \lambda_4)W(1-\xi) - \sigma \qquad (3.6)$$

(W is the energy of the system).

Using in (3.6) the expression for the derivative $\gamma^{\cdot} = \xi^{\cdot}d\gamma/d\xi$ and taking (3.5) into account, after integrating the equation, we find the following amplitude-versus-phase dependence for the vibrations subjected to phase and amplitude modulation /1/:

$$\cos 2\gamma = \frac{4\sigma}{\Gamma_2W}\frac{1}{(1-\xi)} - \frac{2(\lambda_1-\lambda_3)}{\Gamma_2}\frac{\xi}{(1-\xi)} - \frac{2(\lambda_2-\lambda_4)}{\Gamma_1}\frac{(1-\xi)}{\xi} + \frac{G}{\xi(1-\xi)} \qquad (3.7)$$

where G is an arbitrary constant, which can be found from the initial conditions.

Relation (3.7) determines the amplitude-versus-phase dependence for the vibrations subjected both to phase and amplitude modulation and described by Eqs.(3.3).

To find the time dependence of the amplitudes and phases, it is necessary to substitute (3.7) into Eq.(3.5), from which one can obtain the dependence of ξ on T_2 by integrating the equation. Using the function $\xi(T_2)$ in (3.4) and (3.5), we can find $a_\alpha(T_2)$ and $\varphi_\alpha(T_2)$.

4. As an example we consider the free vibrations of the following suspension bridges: "The Golden Gate" in San Francisco, whose length is 1281 m (Table 1) and "Vincent Thomas" in Loś Angeles, which is 557.5 m long (Table 2). In Tables 1 and 2 the resonance frequencies of these bridges and the frequencies of vertical and twisting vibrations, which are close one to another, are listed along with the corresponding modes. The twisting modes at resonance frequencies are not given in Table 1, since in all cases except the third one they are identical with the twisting modes for the characteristic frequencies which are close to one another. In the third case the character of the behaviour of the twisting mode with the frequency $\Omega_{03}^{as} = 2.4$ resembles the character of the behaviour of the vertical mode with frequency ω_{03}^{as} given in Table 2. We note that there are no resonance modes in the frequency spectrum of the "Vincent Thomas" bridge /1/.

Table 1

Similar characteristic frequencies rad/s		Resonance frequencies rad/s
$\omega_{03}^{s} = 1.60$	$\Omega_{01}^{s} = 1.33$	$\omega_{06}^{s} = 2.62$
$\omega_{04}^{s} = 1.80$	$\Omega_{02}^{s} = 1.63$	$\omega_{07}^{s} = 3.75$
$\omega_{05}^{s} = 2.61$	$\Omega_{03}^{s} = 2.67$	$\omega_{08}^{s} = 4.66$
$\omega_{02}^{as} = 1.14$	$\Omega_{01}^{as} = 1.17$	$\omega_{06}^{as} = 3.15$
$\omega_{04}^{as} = 2.11$	$\Omega_{02}^{as} = 2.19$	$\omega_{07}^{as} = 4.41$

Table 2

Similar characteristic frequencies rad/s	
$\omega_{04}^{s} = 2.88$	$\Omega_{01}^{s} = 2.99$
$\omega_{05}^{s} = 5.07$	$\Omega_{02}^{s} = 6.25$
$\omega_{06}^{s} = 6.92$	$\Omega_{04}^{s} = 6.64$
$\omega_{03}^{as} = 3.46$	$\Omega_{01}^{as} = 4.10$
$\omega_{04}^{as} = 6.32$	$\Omega_{02}^{as} = 6.23$

In the case of resonance interaction of the sixth symmetric mode of vertical vibrations with the first symmetric mode of twisting vibrations (Fig.2, (a-e) and also in the cases where the second skew-symmetric form of vertical vibrations is close to the first skew-symmetric form of twisting vibrations (Fig.3, (a-e) and the fourth symmetric form of vertical vibrations is close to the second symmetric form of twisting vibrations (Fig.4, (a-e) the graphs of the dependence on T_1 and T_2 of the quantities ξ (the solid lines) and $1-\xi$ (the broken lines), which are proportional to the squares of the amplitudes a_1 and a_2, respectively, are constructed on the basis of the data in Table 1 for various initial values ξ_0 $(\cos\gamma_0 = 1)$.

From the graphs one can see that in the case of internal resonance for $\xi_0 = 1/3$ (Fig.2c) and in the case of similar frequencies $\omega_{02}^{as} \approx \Omega_{01}^{as}$ for $\xi_0 = 0.174$ (Fig.3b) steady-state conditions are realized, which can be characterized by constant amplitudes (frequency modulation). In both cases the direction of energy transmission changes to the opposite one at the corresponding instants of time as ξ_0 passes through the steady-state mode. Despite the fact that the behaviour of the amplitudes is qualitatively similar in these two cases, considerable quantitative differences can be observed. Namely, in the case of internal resonance the energy transmission from one of the subsystems to the other is more intensive (the peak-to-peak scope of the amplitudes is larger) and faster (the time scale is smaller) than in the case of similar frequencies.

For $\omega_{04}{}^{s} \approx \Omega_{02}{}^{s}$ (Fig.4, (a-e) there are no steady-state conditions (there is no frequency modulation), and so there is no initial value ξ_0 such that the direction of energy transmission changes as one passes through this value. It is seen that this case differs both qualitatively and quantitatively from the case of internal resonance.

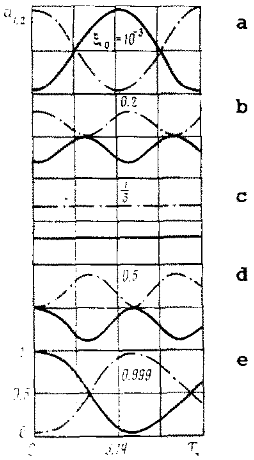

Fig.2

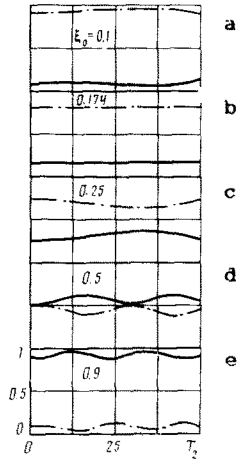

Fig.3

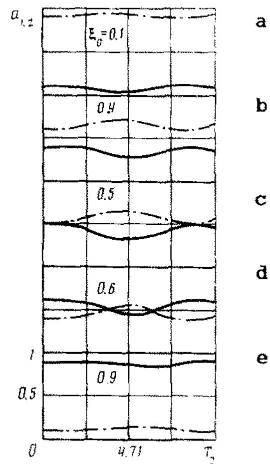

Fig.4

We shall analyse the stationary regimes $a_\alpha = \mathrm{const}$ both in the case of internal resonance and in the case of similar frequencies. It follows from (2.2) that $\gamma = 0$ and $\varphi_\alpha{}' = \mathrm{const}$.

Setting $\varphi_2{}' = \omega\ (\varphi_1{}' = 2\omega)$ and $a_2 = a$, we find from (2.2) the relation

$$(2\omega + \sigma)(\omega + \sigma) - {}^1/_4 a_{12} a_{22} \omega_0^{-1} \Omega_0^{-1} a^2 = 0 \tag{4.1}$$

between the amplitude, the frequency, and the detuning. Next, using (1.5) and (1.9), we obtain the following expressions for the generalized displacements corresponding to the amplitude-versus-phase dependence (4.1):

$$x_1(t) = 4\Omega_0 a_{12}^{-1} \varepsilon (\omega + \sigma) \cos 2\chi t + 8/3 a_{11} \Omega_0^2 a_{12}^{-2} \omega_0^{-2} (\omega +$$
$$\sigma)^2 \varepsilon^2 \cos 4\chi t - 2\varepsilon^2 a_{12}^{-2} \omega_0^{-2} [4a_{11}\Omega_0^2 (\omega + \sigma)^2 + a_{12}^2 a_{22} a^2] + O(\varepsilon^3)$$
$$x_2(t) = 2a\varepsilon \cos \chi t + {}^1/_2 a\varepsilon^2 (\omega + \sigma) \Omega_0^{-1} \cos 3\chi t + O(\varepsilon^3)$$
$$\chi = {}^1/_2 [\omega_0 + (2\omega + \sigma)\varepsilon] = \Omega_0 + (\omega + \sigma)\varepsilon$$

where χ is the frequency of the non-linear vibrations subject to phase modulation only.

The dependence of the square of the amplitude a on the magnitude of detuning σ computed from (4.1) for various values of ω is shown in Fig.5. The numerical values of ω are written next to the corresponding curves and $a_{12}a_{22}(4\omega_0\Omega_0)^{-1}$ is taken to be equal to one. One can see that the character of the amplitude-versus-frequency dependence is determined by the degree of stiffness or softness of the frequency characteristics of the combined suspension system (a soft characteristic corresponds to a negative value of ω and a stiff characteristic corresponds to a positive value).

For $a_\alpha = \mathrm{const}$, we find from (3.3) the relations

$$\begin{aligned} \lambda_1 a_1^2 + (\lambda_2 - {}^1/_4\Gamma_1) a_2^2 &= \varkappa \\ (\lambda_3 - {}^1/_4\Gamma_2) a_1^2 + \lambda_4 a_2^2 &= \varkappa + \sigma \end{aligned} \tag{4.2}$$

between the amplitudes a_1 and a_2, the frequency $\varkappa = \varphi_1{}' = \varphi_2{}'$, and the detuning σ.

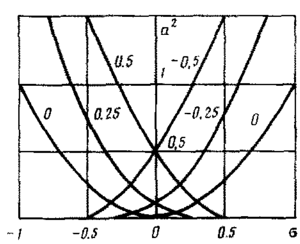

Fig.5

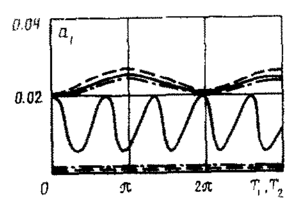

Fig.6

Table 3

Bridge	Symmetric characteristic forms			Skew-symmetric characteristic forms	
"Golden Gate"	$\omega_{03}^{s} = 6.63\mathrm{x}10^{-4}$	$\omega_{04}^{s} = 7.50\mathrm{x}10^{-4}$	$\omega_{05}^{s} = 10.82\mathrm{x}10^{-4}$	$\omega_{02}^{as} = 4.73\mathrm{x}10^{-4}$	$\omega_{04}^{as} = 8.74\mathrm{x}10^{-4}$
	$-1.52\mathrm{x}10^{-5} \leqslant \varepsilon^2\varkappa \leqslant +\infty$	no frequency modulation		$2.33\mathrm{x}10^{-5} \leqslant \varepsilon^2\varkappa \leqslant 2.34\mathrm{x}10^{-5}$	$8.98\mathrm{x}10^{-5} \leqslant \varepsilon^2\varkappa \leqslant 8.99\mathrm{x}10^{-5}$
	$\Omega_{01}^{s} = 5.51\mathrm{x}10^{-4}$	$\Omega_{02}^{s} = 7.58\mathrm{x}10^{-4}$	$\Omega_{03}^{s} = 10.81\mathrm{x}10^{-4}$	$\Omega_{01}^{as} = 4.85\mathrm{x}10^{-4}$	$\Omega_{02}^{as} = 9.07\mathrm{x}10^{-4}$
"Vincent Thomas"	$\omega_{04}^{s} = 6.66\mathrm{x}10^{-4}$	$\omega_{05}^{s} = 11.73\mathrm{x}10^{-4}$	$\omega_{06}^{s} = 16.01\mathrm{x}10^{-4}$	$\Omega_{03}^{as} = 8.01\mathrm{x}10^{-4}$	$\omega_{01}^{as} = 16.01\mathrm{x}10^{-4}$
	$0.37\mathrm{x}10^{-4} \leqslant \varepsilon^2\varkappa \leqslant +\infty$	$6.27\mathrm{x}10^{-4} \leqslant \varepsilon^2\varkappa \leqslant +\infty$	no frequency modulation	$2.496\mathrm{x}10^{-4} \leqslant \varepsilon^2\varkappa \leqslant 2.498\mathrm{x}10^{-4}$	$-3.427\mathrm{x}10^{-4} \leqslant \varepsilon^2\varkappa \leqslant -3.428\mathrm{x}10^{-1}$
	$\Omega_{01}^{s} = 6.92\mathrm{x}10^{-4}$	$\Omega_{02}^{s} = 14.46\mathrm{x}10^{-4}$	$\Omega_{04}^{s} = 15.36\mathrm{x}10^{-4}$	$\Omega_{01}^{as} = 9.48\mathrm{x}10^{-4}$	$\Omega_{02}^{as} = 14.42\mathrm{x}10^{-4}$

Using formulae (1.5), (3.1), (1.9), and Eqs.(1.10), we obtain the following expressions for the generalized displacements corresponding to the amplitude-versus-frequency relations (4.2):

$$
\begin{aligned}
x_1(t) &= 2\varepsilon a_1 \cos \beta t + 2(\varepsilon a_1)^2 (-1 + {}^1/_3 \cos 2\beta t)\, a_{11}\omega_0^{-2} + \\
&\quad 2(\varepsilon a_2)^2 [-1 + {}^1/_3 \cos 2(\beta - \sigma\varepsilon^2)t]\, a_{22}\omega_0^{-2} + O(\varepsilon^3) \\
x_2(t) &= 2\varepsilon a_2 \cos \beta t + 2\varepsilon^2 a_1 a_2 (-1 + {}^1/_3 \cos 2\beta t)\, a_{12}\Omega_0^{-2} + O(\varepsilon^3) \\
\beta &= \omega_0 + \varkappa\varepsilon^2 = \Omega_0 + \varkappa\varepsilon^2 + \sigma\varepsilon^2
\end{aligned} \tag{4.3}
$$

where β is the frequency of non-linear vibrations, which is the same for the vertical vibrations and the bending and twisting vibrations.

The results of the analysis of the vibrations subject to frequency modulation only based on the data in Tables 1 and 2 are collected in Table 3. It can be seen that the domain of existence of the solution determined by (4.2) and (4.3) is reduced to a point for every skew-symmetric characteristic form of vibrations (such vibrations are possible for one value of the frequency $\varkappa$), and either extends to infinity or disappears completely for every symmetric form of characteristic vibrations.

Therefore, in the case of similar frequencies the non-linear vibrations (4.3) subject to frequency modulation can be realized for the given values ω_{0n}, Ω_{0m} and σ only, while, as can be seen from (4.1), in the case of internal resonance such vibrations can be excited for any values of ω_{0n}, Ω_{0m} and σ. These results are consistent with the graphs in Figs.2-4.

Finally, we mention that the regime of vibrations in the case of similar frequencies turns out to be more stable under variations of the level of mistuning than the regime of vibrations in the case of internal resonance. This is evident from the graphs of the time dependence of the amplitude function envelopes for the following three values of detuning: $\varepsilon^2\sigma = -0.5\times10^{-4}$ (the broken line), $\sigma = 0$ (the solid line), and $\varepsilon^2\sigma = 0.5\times10^{-4}$ (the dash-dot line), which are given in Fig.6 for the case of internal resonance $(a_1(T_1) < 0.02)$ and the case of similar frequencies $(a_1(T_2) \geqslant 0.02)$. One can see that even a slight violation of the resonance condition $\omega_0 = 2\Omega_0$ results in the maxima of the amplitude functions being immediately reduced to zero.

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Translated by T.Z.

PMM U.S.S.R.,Vol.54,No.6,pp. 832-837,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

STRESSES IN ELASTIC CONICAL TUBES OF TRANSVERSELY ISOTROPIC MATERIALS WITH SPHERICAL ANISOTROPIES UNDER TEMPERATURE AND FORCE LOADINGS*

I.V. PANFEROV

Analytic solutions are proposed for a number of new problems on determining the state of stress of a transversely-isotropic hollow cone with spherical anisotropy. An exact solution of the problem of the axisymmetric deformation of a long conical tube (or continuous cone) from an elastic transversely-isotropic material with spherical anisotropy subjected to an axial force is obtained in a spherical coordinate system R, φ, θ; the material axis of symmetry is directed along the spherical radius R. A rigorous solution is given of the problem of the uniform heating of a conical tube of transversely-isotropic material with spherical anisotropy for particular values of Poisson's ratios; the material axis of symmetry is directed along the θ-axis. For arbitrary Poisson's ratios an asymptotic solution is found for the temperature problem for a tube with small conicity.

**Prikl.Matem.Mekhan.*,54,6,1012-1016,1990